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REDUCTION OF THE DIFFERENTIAL EQUATIONS OF NONHOLONOMIC MECHANICS TO LAGRANGE FORM

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We consider the problem of the equivalence of a certain system of ordinary differential equations to a system of Lagrange equations. Wherever we do not expressly say so, we have in mind sationary nonholonomic Chaplygin systems with linear constraints. The equations of motion of non-holonomic systems in the Routh form, Chaplygin in appearance, differ from the Lagrange equations of the second kind in the presence of additional terms (constraint reactions, nonholonomic terms). This fact hinders the extension of integration methods of equations of motion of holonomic systems to nonholonomic ones. The few attempts [1, 2] to seek general methods for integrating the equations of nonholonomic mechanics were reduced to the transformation of the equations of motion to Lagrange form [3]. The equations of motion of nonholonomic systems have the form of Lagrange equations [1, 4] only in exceptional cases.

The problem of determining the conditions which guarantee the equivalence of a given system of differential equations

$$F_j(q_1^{\cdot\cdot}, \ldots, q_n^{\cdot\cdot}, q_1^{\cdot}, \ldots, q_n^{\cdot}, q_1, \ldots, q_n, t) = 0 \qquad (j = 1, \ldots, n) \tag{0.1}$$

with the Lagrange system

$$L_j(\theta) = 0 \qquad (j = 1, \ldots, n) \qquad \left(L_j = \frac{d}{dt}\frac{\partial}{\partial q_j^{\cdot}} - \frac{\partial}{\partial q_j} \right) \tag{0.2}$$

where θ is a certain function of $q_1^{\cdot}, \ldots, q_n^{\cdot}, q_1, \ldots, q_n, t$, is a familiar one. Necessary and sufficient conditions (the Helmholtz conditions) were obtained in [5 - 7] on whose basis we can determine from the appearance of Eqs. (0.1) whether each of these equations individually is a Lagrange equation relative to the function θ called the Helmholtz kinetic potential. It should be noted that the Helmholtz conditions applied by Chaplygin are usually not fulfilled for Routh equations [8, 9], nevertheless, in some cases, by combining these equations they can be replaced by an equivalent Lagrange system [8]. A theorem was proven in [10] that the equations of motion of a mechanical system with linear nonintegrable constraints

$$w_i = q_i^{\cdot} + \sum_{s=k+1}^{n} a_{is}(q_1, \ldots, q_n, t)\, q_s^{\cdot} + a_i(q_1, \ldots, q_n, t) = 0 \quad (i = 1, \ldots, k < n)$$

are not equivalent to the system of Eqs. (0.2), where

$$\theta = F + \sum_{i=1}^{k} \Lambda_i w_i, \quad F = \frac{1}{2} \sum_{j,h=1}^{n} b_{jh} q_j^{\cdot} q_h^{\cdot} + \sum_{j=1}^{n} c_j q_j^{\cdot} + P$$

(b_{jh}, c_j, P are arbitrary functions of $q_1, \ldots, q_n, t$), Λ_i are certain functions chosen from the condition that the equations for the extremals of the functional $\int_{t_0}^{t_1} \theta dt$ have the form (0.2) (a conditional variational problem with fixed end-points). But for certain nonholonomic mechanical systems Eqs. (0.2) may be valid [11] when there exists a particular solution of the so-called "equations of equivalence". It was shown in [11] how to set up the equations of equivalence, but the equations themselves were not cited and they were not analyzed. If the equations of motion of a nonholonomic system are solved relative to acceleration, then the right-hand sides of the equations obtained are certain quadratic forms in the velocities. Below, for a system of two such equations with right-hand sides which are arbitrary homogeneous quadratic forms in the velocities, we prove a theorem which, with the aid only of differentiation and algebraic manipulations, allow us to ascertain whether we can obtain the given equations from some system of Lagrange equations by means of a solution relative to the second derivatives, if the kinetic potential is an arbitrary nondegenerate homogeneous quadratic form in the velocities.

1. Consider the two systems of equations

$$q_k^{\cdot\cdot} = F_k \qquad (k = 1, \ldots, m) \tag{1.1}$$

$$L_r(\theta) = 0 \qquad (r = 1, \ldots, m) \tag{1.2}$$

in which

$$F_k = \frac{1}{2} \sum_{i,r=1}^{m} f_{ri}^{k}(q_1, \ldots, q_m) q_r^{\cdot} q_i^{\cdot} \qquad (f_{ri}^{k} = f_{ir}^{k})$$

$$\theta = \frac{1}{2} \sum_{i,j=1}^{m} c_{ij}(q_1, \ldots, q_m) q_i^{\cdot} q_j^{\cdot} \qquad (c_{ij} = c_{ji})$$

where

$$\det \| c_{ij} \| \neq 0 \tag{1.3}$$

For these systems to be equivalent it is necessary [12] that the function θ satisfy the collection of partial differential equations

$$\sum_{k=1}^{m} \frac{\partial^2 \theta}{\partial q_r^{\cdot} \partial q_k^{\cdot}} F_k + \sum_{k=1}^{m} \frac{\partial^2 \theta}{\partial q_r^{\cdot} \partial q_k} q_k^{\cdot} - \frac{\partial \theta}{\partial q_r} = 0 \quad (r = 1, \ldots, m) \tag{1.4}$$

where the quantities $q_1, \ldots, q_m, q_1^{\cdot}, \ldots, q_m^{\cdot}$ are treated as independent variables. The values of $q_1^{\cdot}, \ldots, q_m^{\cdot}$ can be chosen arbitrarily, therefore the coefficients of the quadratic forms must vanish identically

$$\omega_{rij} = \sum_{k=1}^{m} c_{rk} f_{ij}^{k} + \frac{\partial c_{ri}}{\partial q_j} + \frac{\partial c_{rj}}{\partial q_i} - \frac{\partial c_{ij}}{\partial q_r} = 0 \quad (r, i, j = 1, \ldots, m) \tag{1.5}$$

In order to simplify the system of equations (1.5) we replace every equation $\omega_{rij} = 0$ by the equation $\varkappa_{rij} = {}^1/_2 (\omega_{irj} + \omega_{jir}) = 0$, and we obtain

$$\varkappa_{rij} = \frac{1}{2}\sum_{k=1}^{m} (c_{kj}f_{ri}^{k} + c_{ki}f_{rj}^{k}) + \frac{\partial c_{ij}}{\partial q_r} = 0 \quad (r, i, j = 1, \ldots, m) \tag{1.6}$$

Conversely, $\omega_{ijr} = \varkappa_{rij} - \varkappa_{ijr} + \varkappa_{jri}$. Relations (1.6), in which the functions f_{ri}^{k} are assumed specified, represent a system of $^1/_2\, m^2\,(m+1)$ partial differential equations in the $^1/_2\, m\,(m+1)$ of unknown functions c_{ij}. We are interested in the conditions when this system admits of a particular solution satisfying inequality (1.3). Then by virtue of Eqs. (1.4), the system of equations (1.1) can be replaced by an equivalent system (1.2).

2. We investigate the case $m = 2$. Let

$$q_1^{\cdot\cdot} = {}^1/_2\, a q_1^{\cdot 2} + c q_1^{\cdot} q_2^{\cdot} + {}^1/_2\, b q_2^{\cdot 2}, \quad q_2^{\cdot\cdot} = {}^1/_2\, \alpha q_1^{\cdot 2} + \gamma q_1^{\cdot} q_2^{\cdot} + {}^1/_2\, \beta q_2^{\cdot 2} \tag{2.1}$$

where a, b, c, α, β, γ are functions of q_1 and q_2. We denote

$$\theta = {}^1/_2 X(q_1, q_2)\, q_1^{\cdot 2} + Z(q_1, q_2)\, q_1^{\cdot} q_2^{\cdot} + {}^1/_2 Y(q_1, q_2)\, q_2^{\cdot 2} \tag{2.2}$$

Condition (1.3) takes the form

$$XY - Z^2 \neq 0 \tag{2.3}$$

Equations (1.6) are written in the form of the equations

$$\partial X/\partial q_1 + aX + \alpha Z = 0, \quad \partial X/\partial q_2 + cX + \gamma Z = 0 \tag{2.4}$$

$$\partial Y/\partial q_1 + cZ + \gamma Y = 0, \quad \partial Y/\partial q_2 + bZ + \beta Y = 0 \tag{2.5}$$

$$\partial Z/\partial q_1 + {}^1/_2\, [cX + (a + \gamma) Z + \alpha Y] = 0 \tag{2.6}$$

$$\partial Z/\partial q_2 + {}^1/_2\, [bX + (c + \beta) Z + \gamma Y] = 0$$

which can be written more concisely with the aid of differentials

$$dX + (aX + \alpha Z)\, dq_1 + (cX + \gamma Z)\, dq_2 = 0 \tag{2.7}$$

$$dY + (cZ + \gamma Y)\, dq_1 + (bZ + \beta Y)\, dq_2 = 0 \tag{2.8}$$

$$dZ + {}^1/_2\, [cX + (a + \gamma) Z + \alpha Y]\, dq_1 + {}^1/_2\, [bx + (c + \beta) Z + \gamma Y] dq_2 = 0 \tag{2.9}$$

We introduce into consideration functions of the coefficients of Eqs. (2.1)

$$\begin{aligned} A_1 &= \frac{\partial a}{\partial q_2} - \frac{\partial c}{\partial q_1} - \frac{1}{2}(\alpha b - \gamma c) \\ A_2 &= \frac{\partial \alpha}{\partial q_2} - \frac{\partial \gamma}{\partial q_1} + \frac{1}{2}(\alpha c - \gamma a + \gamma^2 - \alpha\beta) \\ B_1 &= \frac{\partial \gamma}{\partial q_2} - \frac{\partial \beta}{\partial q_1} + \frac{1}{2}(\alpha b - \gamma c) \\ B_2 &= \frac{\partial c}{\partial q_2} - \frac{\partial b}{\partial q_1} - \frac{1}{2}(\gamma b - \beta c + c^2 - ab) \end{aligned} \tag{2.10}$$

The following theorem is valid.

Theorem. For the system of equations (2.1) to be equivalent to any Lagrange system $L_1(\theta) = L_2(\theta) = 0$ (θ is some function (2.2) satisfying inequality (2.3)) it is necessary and sufficient to fulfill one of the tnree conditions

$$1)\ A_1 = B_1 = A_2 = B_2 = 0$$

2) $A_1 = B_1 = 0,\ A_2 B_2 \neq 0,\ A_2 b - B_2\gamma = 0,\ A_2 c - B_2\alpha = 0$

$B_2/A_2\, e^{\psi-\varphi} = \text{const}\ (d\varphi = a dq_1 + c dq_2,\ d\psi = \gamma dq_1 + \beta dq_2)$

3) $A_1 = -B_1 \neq 0,\ A_2B_2 + A_1^2 \neq 0$

$$\frac{\partial}{\partial q_2}\left(-c\frac{A_2}{A_1} + a + \gamma + \alpha\frac{B_2}{A_1}\right) - \frac{\partial}{\partial q_1}\left(-b\frac{A_2}{A_1} + c + \beta + \gamma\frac{B_2}{A_1}\right) = 0$$

$$\frac{1}{2}\frac{A_2}{A_1}\left(-c\frac{A_2}{A_1} + a + \gamma + \alpha\frac{B_2}{A_1}\right) - \frac{\partial}{\partial q_1}\left(\frac{A_2}{A_1}\right) - a\frac{A_2}{A_1} + \alpha = 0$$

$$\frac{1}{2}\frac{A_2}{A_1}\left(-b\frac{A_2}{A_1} + c + \beta + \gamma\frac{B_2}{A_1}\right) - \frac{\partial}{\partial q_2}\left(\frac{A_2}{A_1}\right) - c\frac{A_2}{A_1} + \gamma = 0$$

$$-\frac{1}{2}\frac{B_2}{A_1}\left(-c\frac{A_2}{A_1} + a + \gamma + \alpha\frac{B_2}{A_1}\right) + \frac{\partial}{\partial q_1}\left(\frac{B_2}{A_1}\right) + \gamma\frac{B_2}{A_1} + c = 0$$

$$-\frac{1}{2}\frac{B_2}{A_1}\left(-b\frac{A_2}{A_1} + c + \beta + \gamma\frac{B_2}{A_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{B_2}{A_1}\right) + \beta\frac{B_2}{A_1} + b = 0$$

Proof. Let us assume that the system of equations (2.4) - (2.6) has a certain solution X, Y, Z which satisfies inequality (2.3). We consider $d\,(Z^2 - XY)$ relative to (2.7) - (2.9)

$$d\,(Z^2 - XY) = -(Z^2 - XY)\,[(a + \gamma)\,dq_1 + (c + \beta)\,dq_2]$$

Consequently (see (2.10)),

$$A_1 + B_1 = 0 \tag{2.11}$$

Computing from Eqs. (2.4) - (2.6) the expressions

$$\frac{\partial^2 X}{\partial q_2 \partial q_1} - \frac{\partial^2 X}{\partial q_1 \partial q_2} = 0, \quad \frac{\partial^2 Y}{\partial q_2 \partial q_1} - \frac{\partial^2 Y}{\partial q_1 \partial q_2} = 0, \quad \frac{\partial^2 Z}{\partial q_2 \partial q_1} - \frac{\partial^2 Z}{\partial q_1 \partial q_2} = 0$$

and using relation (2.11), we obtain the system of equations

$$A_1 X + A_2 Z = 0, \qquad -A_1 Y + B_2 Z = 0, \qquad B_2 X + A_2 Y = 0 \tag{2.12}$$

which the functions X, Y, Z must satisfy identically. Two cases are possible: $A_1 = A_2 = B_2 = 0$ or $A_1^2 + A_2^2 + B_2^2 \neq 0$.

In the first case (Condition (1) of the theorem) the system of equations (2.7) – (2.9) is a system in total differentials. We can be convinced of this, for example, with the aid of Frobenius' theorem. Consequently, there exist three independent integrals of Eqs. (2.7) – (2.9). The constants in these integrals can always be chosen so as to fulfil inequality (2.3), otherwise the integrals would not be independent.

In the second case there may be two subcases $A_1 = 0$ or $A_1 \neq 0$. We consider each of them separately. Suppose that $A_1 = 0$. Then, from Eqs. (2.12) and condition (2.3) we obtain $A_2B_2 \neq 0$. Consequently, $Z = 0$ (see (2.12)) and for the fulfillment of inequality (2.3) it is necessary that the equations (see (2.6), (2.12))

$$cX + \alpha Y = 0, \qquad bX + \gamma Y = 0, \qquad B_2 X + A_2 Y = 0 \tag{2.13}$$

be satisfied with nonzero values of X and Y. Therefore, it is necessary that $\alpha b - c\gamma = 0$. But then, taking (2.11) into consideration, we see (see (2.10)) that the identities

$$\frac{\partial a}{\partial q_2} - \frac{\partial c}{\partial q_1} = 0, \qquad \frac{\partial \gamma}{\partial q_2} - \frac{\partial \beta}{\partial q_1} = 0$$

are valid in the subcase being considered. Since $Z = 0$, from (2.7) and (2.8) we obtain

$$X = C_X e^{-\varphi}, \quad Y = C_Y e^{-\psi} \qquad (C_X, C_Y = \text{const})$$

By virtue of (2.13) it is necessary that

$$-\frac{B_2}{A_2}e^{\psi-\varphi}=C_Y/C_X$$

Conversely, if Condition (2) of the theorem is fulfilled, then Eqs. (2.7) – (2.9) are satisfied by the functions

$$X=e^{-\varphi},\quad Y=B_2/A_2\,e^{-\varphi},\quad Z=0,\quad XY-Z^2\neq 0$$

Let us consider the second subcase: $A_1\neq 0$. From the first two equations in (2.12) follow

$$X=-\frac{A_2}{A_1}Z,\qquad Y=\frac{B_2}{A_1}Z \tag{2.14}$$

while the third equation in (2.12) is fulfilled automatically. Inequality (2.3) is fulfilled if $Z\neq 0$ and $A_2B_2+A_1^2\neq 0$. Substituting the functions (2.14) into Eq. (2.9) we obtain

$$dZ+Z\left[\frac{1}{2}\left(-c\frac{A_2}{A_1}+a+\gamma+\alpha\frac{B_2}{A_1}\right)dq_1+\frac{1}{2}\left(-b\frac{A_2}{A_1}+c+\beta+\gamma\frac{B_2}{A_1}\right)dq_2\right]=0$$

This equation has a nontrivial solution if and only if the expression within the brackets is an exact differential of some function $\mu(q_1,q_2)$. Then $Z=C_Ze^{-\mu}$ (C_Z is an arbitrary constant). The last four identities in Condition (3) of the theorem are necessary and sufficient for Eqs. (2.4) and (2.5) to be satisfied by the functions

$$X=-\frac{A_2}{A_1}e^{-\mu},\qquad Y=\frac{B_2}{A_1}e^{-\mu},\qquad Z=e^{-\mu}$$

The theorem is proven. It is not difficult to obtain a corresponding generalization to the case when forces enter into the equations of motion.

3. The preceding theorem admits of a simple formulation if $a=b=c=0$ or $\alpha=\beta=\gamma=0$. For example, let

$$T=\frac{Iq_1^{\cdot 2}}{2}+\frac{1}{2}\sum_{i,h=2}^{n}b_{jh}(q_2)\,q_j^{\cdot}q_h^{\cdot}\qquad (I=\text{const})$$

be the kinetic energy of a nonholonomic system with the constraints

$$q_j^{\cdot}=f_j(q_1,q_2)\,q_2^{\cdot}\qquad (j=3,\ldots,n)$$

Obviously, the Chaplygin equations, solved relative to $q_1^{\cdot\cdot}$, $q_2^{\cdot\cdot}$, have in the case of the inertial motion of the system the form (2.1) where $a=b=c=0$. From the theorem in Sect. 2 it follows that for the equivalence of these equations to a Lagrange system it is necessary and sufficient that

$$\frac{\partial\gamma}{\partial q_2}-\frac{\partial\beta}{\partial q_1}=0,\qquad \frac{\partial\alpha}{\partial q_2}-\frac{\partial\gamma}{\partial q_1}+\frac{1}{2}(\gamma^2-\alpha\beta)=0 \tag{3.1}$$

As an example let us consider the well-known Chaplygin problem of the nonholonomic inertial motion of a body along a horizontal plane [1]. A rigid body rests on a plane at three points, two of which are freely sliding legs while the third is the point A of contact of a knife-edged caster rigidly attached to the body. The caster cannot slide in a direction perpendicular to its plane. Let us assume also that the center of gravity of the rigid body is located in the vertical plane passing through the point A perpendicular to the caster's plane. Let q_1 be the angle of rotation of the body around a vertical axis, and let q_2, q_3 be the Cartesian coordinates of point A on the horizontal plane. The nonholonomic condition is expressed by the equation $q_3 = q_2 \operatorname{tg} q_1$ and the kinetic energy of the body by

$$T = \frac{m}{2}\left[(q_2^{\cdot} - q_1^{\cdot} k \cos q_1)^2 + (q_3^{\cdot} - q_1^{\cdot} k \sin q_1)^2 + l q_1^{\cdot 2}\right] \quad (m,\, k,\, l = \text{const})$$

Solving the Chaplygin equations relative to the accelerations we obtain

$$q_1^{\cdot\cdot} = 0, \qquad q_2^{\cdot\cdot} = -q_1^{\cdot} q_2^{\cdot} \operatorname{tg} q_1 \tag{3.2}$$

Consequently, $a = b = c = 0$, $\alpha = \beta = 0$, $\gamma = -\operatorname{tg} q_1$. Condition (3.1) is not fulfilled. Chaplygin showed [1] that if in the case being considered we introduce a nonholonomic coordinate s, namely the length of the trajectory arc of point A, then in the variables s and q_1 the equations of motion take the form of the Lagrange equations $s^{\cdot\cdot} = 0$, $q_1 = 0$ (see [13]). From the theorem that was proved it follows that it is impossible to write Eqs. (3.2) in Lagrange coordinates in the form of the equations $L_1(\theta) = L_2(\theta) = 0$ whatever be the function (2.2) satisfying the inequality (2.3).

4. Let us indicate one condition under whose fulfillment the integration of the Chaplygin equations can be replaced by the integration of the Lagrange equations. Let $q_1, \ldots, q_n$ be the coordinates of a mechanical system whose motion is restricted by the constraints

$$q_i^{\cdot} = \sum_{s=k+1}^{n} a_{is}(q_{k+1}, \ldots, q_n)\, q_s^{\cdot} \qquad (i = 1, \ldots, k) \tag{4.1}$$

Assume that the Lagrangian function of the system has the form

$$L = \theta_1(q_1^{\cdot}, \ldots, q_k^{\cdot}) + \theta_2(q_{k+1}^{\cdot}, \ldots, q_n^{\cdot}, q_{k+1}, \ldots, q_n) \tag{4.2}$$

We write the equations of motion in the Routh form

$$L_i(\theta_1) = \lambda_i \qquad (i = 1, \ldots, k) \tag{4.3}$$

$$L_s(\theta_2) = R_s, \qquad R_s = -\sum_{i=1}^{k} \lambda_i a_{is} \qquad (s = k+1, \ldots, n) \tag{4.4}$$

Using the constraint equations (4.1) we express the multipliers λ_i as functions of $q^{\cdot}_{k+1}, \ldots, q^{\cdot}_n, q_{k+1}, \ldots, q_n$. Then Eqs. (4.4) can be considered independently of (4.3). For comparison we write down the Chaplygin equations

$$L_s(\theta_1^* + \theta_2) = \Phi_s, \quad \Phi_s = \sum_{i=1}^{k} \left(\frac{\partial \theta_1}{\partial q_i^{\cdot}}\right)^* L_s\left(\sum_{r=k+1}^{n} a_{ir} q_r^{\cdot}\right) \quad (s = k+1, \ldots, n) \tag{4.5}$$

The asterisk denotes the result of eliminating the velocities $q_i^{\cdot}$ $(i = 1, \ldots, k)$ from the corresponding expressions (see (4.1)). Equations (4.4) and (4.5) are equivalent. Therefore, if there were to exist a function $\theta = \theta(q^{\cdot}_{k+1}, \ldots, q_n^{\cdot}, q_{k+1}, \ldots, q_n)$ such that (*)

$$R_s = L_s(\theta) \qquad (s = k+1, \ldots, n) \tag{4.6}$$

by virtue of Eqs. (4.4) and of

$$\det \left\| \frac{\partial^2(\theta_2 - \theta)}{\partial q_s^{\cdot} \partial q_r^{\cdot}} \right\| \neq 0$$

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*) Conditions (4.6) have been obtained by other means by S. O. Titkova (see S. O. Titkova: The rolling of a ball on a rough plane. Candidate's Dissertation, Alma - Ata, Izd. Kazakhs. Gos. Univ., 1970).

then the Chaplygin equations would be equivalent to the Lagrange equations $L_s\,(\theta_2 - -\theta)=0$ $(s = k+1,\ldots,n)$. The function $\theta = 0$, satisfies the stated conditions if

$$R_{k+1} = \ldots = R_n = 0 \qquad (4.7)$$

Condition (4.7) is sufficient for Eqs. (4.5) to be replaceable by the equations $L_s\,(\theta_2) = 0$, $(s = k+1,\ldots,n)$. Obviously, relations (4.7) are fulfilled if $\lambda_1 = \ldots = \lambda_k = 0$. Precisely this case obtains for the inertial motion without sliding of a homogeneous ball along a plane. Here [8]

$$\theta_1 = \frac{m}{2}\,(q_1^{\cdot 2} + q_2^{\cdot 2}), \quad \theta_2 = \frac{ma^2}{5}\,(q_3^{\cdot 2} + q_4^{\cdot 2} + q_5^{\cdot 2} + 2q_4^{\cdot} q_5^{\cdot} \cos q_3)$$

where q_1 and q_2 are the Cartesian coordinates of the point of tangency on the plane, q_3, q_4, q_5 are Euler angles, m is the ball's mass and a is its radius. It can be shown [14] that the reaction of the plane on the ball is directed perpendicularly to the plane. Consequently, reaction forces do not enter into the Routh equations, i.e. $\lambda_1 = \lambda_2 = 0$. The validity of the equations $L_3\,(\theta_2) = L_4\,(\theta_2) = L_5\,(\theta_2) = 0$ has been proved in [8] by direct computations, proceeding from the Chaplygin equations set up for the ball.

Note that relations (4.7) may be fulfilled when $\lambda_1^2 + \ldots + \lambda_k^2 \neq 0$. An example is a homogeneous circular disk rolling under inertia along a rough horizontal plane under the condition that the disk's plane remains vertical during the whole time of motion. Let q_1 and q_2 be the Cartesian coordinates of the point of tangency on the plane, q_3 be the angle of the natural revolution of the disk, q_4 be the angle of rotation around the vertical axis. The constraint equations are the following: $q_1^{\cdot} = -aq_3^{\cdot} \cos q_4$, $q_2 = -- aq_3^{\cdot} \sin q_4$. The Lagrange system has the form (4.2) where

$$\theta_1 = {}^1\!/_2 m\,(q_1^{\cdot 2} + q_2^{\cdot 2}), \quad \theta_2 = {}^1\!/_2 ma^2\,(q_3^{\cdot 2} + {}^1\!/_2 q_4^{\cdot 2})$$

(m is the disk's mass, a is its radius). It is not difficult to be convinced that

$$\lambda_1 = maq_3^{\cdot} q_4^{\cdot} \sin q_4, \qquad \lambda_2 = -\,maq_3^{\cdot} q_4^{\cdot} \cos q_4$$

but $R_3 = R_4 = 0$.

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ON THE STABILITY OF LINEAR SYSTEMS WITH RANDOM PARAMETERS

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We propose an approximate method for the investigation of the stability of systems of linear equations with stationary random coefficients, based on the use of the method of perturbations. The problem is reduced to the investigation of the stability of a system of finite-difference equations whose coefficients are determined by the spectral densities of the random parameters. Stability conditions for systems of linear equations with random coefficients have been considered by many authors [1 - 5]. For systems whose coefficients are Gaussian white noises, exact stability criteria have been obtained [1]. Approximate conditions based on the use of asymptotic methods have been found in a number of papers [3, 4] principally for second-order systems with small stationary perturbations of the parameters. The application of these same methods to higher-order systems leads to complicated calculations.

1. We consider the nth order equation

$$\mathbf{y}^{\cdot} = [C + \mu G(t) - \mu^2 B_1]\mathbf{y} \tag{1.1}$$

Here C is a real $n \times n$ matrix with eigenvalues $\lambda_s = ik_s$ $(s = 1, \ldots, 2r)$, $\operatorname{Re}\lambda_s < 0$ $(s = 2r + 1, \ldots, n)$; we assume that all the k_s are distinct (and, obviously, pairwise opposite). The elements of the matrix $G(t)$ are centered stationary random processes, B_1 is a real matrix, μ is a small parameter.